# Dieudonné modules, part I: a study of primitively generated Hopf algebras

Alan Koch

Agnes Scott College

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# Outline

1. Overview
2. Primitive Elements
3. A Dieudonné Correspondence
4. Applications
   - Classification of Hopf algebras
   - Endomorphisms of Hopf algebras
   - Extensions of Hopf algebras
   - Forms of Hopf algebras
   - Hopf orders
5. Want more?

Let $R$ be an $\mathbb{F}_p$-algebra, for example:

- $R = \mathbb{F}_q, \ q = p^n$
- $R = \mathbb{F}_q[T_1, T_2, \ldots, T_r]$
- $R = \mathbb{F}_q(T_1, T_2, \ldots, T_r)$
- $R = \mathbb{F}_q[[T]]$
- $R = \mathbb{F}_q((T))$.

**Objective.** Classify finite, abelian (commutative, cocommutative) $R$-Hopf algebras of $p$-power rank, free as $R$-modules.

**Approach.** Can we construct a category of modules which are equivalent to this category of Hopf algebras?

> *"It is probably impossible to do so" – A.J. de Jong, describing the geometric statement of this problem.*

$R$ an $\mathbb{F}_p$-algebra.

Let $\mathcal{C}$ be a subcategory of finite, abelian $R$-Hopf algebras of $p$-power rank, free as $R$-modules.

**Objective.** Classify the objects of $\mathcal{C}$ using a category of modules.

This can be done for certain $\mathcal{C}$ using Dieudonné modules.

# Some legal disclaimers

Dieudonné's original works (Lie groups and hyperalgebras over a of characteristic $p > 0$, parts I - VI) focus on abelian Lie groups. He appears to be much more interested in infinite dimensional structures (formal groups, $p$-divisible groups).

Dieudonné modules have typically been used by algebraic geometers, trying to describe geometric objects.

In many ways, the development of Dieudonné module theory is better expressed geometrically, but here ("Omaha") algebraic constructions/explanations will be a priority.

# Assumptions

Throughout $H$ is an $R$ Hopf algebra with comultiplication $\Delta$, count $\varepsilon$ and antipode $\lambda$.

All Hopf algebras are assumed to be finite, abelian, free over $R$ and of $p$-power rank unless otherwise specified.

# Possible readings for this talk

- A. Grothendieck, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7) - Groupes de monodromie en géométrie algébrique. Lecture notes in mathematics 288. Berlin; New York: Springer-Verlag. (Apparently.)
- A.J. de Jong, Finite locally free group schemes in characteristic $p$ and Dieudonné modules, Invent. mathematicae 1993, Volume 114, Issue 1, pp 89–137.
- M. Rapoport, Formal moduli spaces in equal characteristic, https://math.berkeley.edu/ aaron/livetex/fmsiec.pdf.

Listed in decreasing order of the amount of algebraic geometry needed as background.

# Outline

# What is a Hopf algebra?

A structure which is both an algebra and a coalgebra where these structures get along in a compatible way.

Idea: what if we focus on Hopf algebras with a particularly simple coalgebra structure?

Let $H$ be finite (i.e., finitely generated and free) $R$-Hopf algebra. An element $t \in H$ is *primitive* if

$$\Delta(t) = t \otimes 1 + 1 \otimes t.$$

Necessarily, if $t \in P(H)$ then $\varepsilon(t) = 0$ and $\lambda(t) = -t$.

Let $P(H)$ be the set of all primitive elements of $H$.

## Example

Let $H = R[t]/(t^p),\ \Delta(t) = t \otimes 1 + 1 \otimes t.$

**Exercise 1.** Show $P(H) = Rt$ .

## Example

Let $H = R[t]/(t^{p^n}),\ \Delta(t) = t \otimes 1 + 1 \otimes t.$

**Exercise 2.** Show $P(H) = Rt + Rt^p + \cdots + Rt^{p^{n-1}}$.

## Example

Let $H = R[t_1, t_2, \ldots, t_n]/(t_1^p, t_2^p, \ldots, t_n^p),\ \Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i.$

**Exercise 3.** Describe $P(H)$ as an $R$-module.

## Example

Let $H = R\Gamma$, $\Gamma$ an abelian $p$-group (or any finite group).

**Exercise 4.** Describe $P(H)$ as an $R$-module.

## Example

Let $R$ be a domain, and $H = RC_p^* = \mathrm{Hom}_R(RC_p, R)$, $C_p = \langle \sigma \rangle$.
Let

$$t = \sum_{i=1}^{p-1} i\epsilon_i,$$

where $\epsilon_i(\sigma^j) = \delta_{i,j}$.

**Exercise 5.** Show that $H = R[t]/(t^p - t)$, and $P(H) = Rt$.

Properties of $P(H)$:

1. $P(H) \cap R = 0$.
2. $t, u \in P(H) \Rightarrow t + u \in P(H)$.
3. $t \in P(H),\ r \in R \Rightarrow rt \in P(H)$.
4. $P(H)$ is an $R$-submodule of $H$.
5. If $R$ is a PID, then $P(H)$ is free over $R$.
6. $t \in P(H) \Rightarrow t^p \in P(H)$.

**Exercise 6.** Prove any subset of the above.

Consider

$$\mathrm{Hom}_{R\text{-Hopf}}(R[x], H)$$

where $x \in R[x]$ is primitive.

There is a map $\mathrm{Hom}_{R\text{-Hopf}}(R[x], H) \to P(H)$ given by

$$f \mapsto f(x)$$

which is an isomorphism.

Define a map $F : \mathrm{Hom}_{R\text{-Hopf}}(R[x], H) \to \mathrm{Hom}_{R\text{-Hopf}}(R[x], H)$ by

$$F(f)(x) = f(x^p) = (f(x))^p.$$

This induces a map $t \mapsto t^p$ in $P(H)$, which we will also denote $F$ and call the *Frobenius map*.

Note $F(rx) = r^p F(x)$ for all $r \in R$.

### Example

Let $H = R[t]/(t^p),\ \Delta(t) = t \otimes 1 + 1 \otimes t$.
Then $P(H) = Rt$ and $F(t) = 0$.

### Example

Let $H = R[t]/(t^{p^n}),\ \Delta(t) = t \otimes 1 + 1 \otimes t$.
Then $P(H) = Rt + Rt^p + \cdots + Rt^{p^{n-1}}$ and $F(t^{p^i}) = t^{p^{i+1}}$.

### Example

Let $H = RC_p^* = R[t]/(t^p - t),\ \Delta(t) = t \otimes 1 + 1 \otimes t$.
Then $P(H) = Rt$ and $F(t) = t^p = t$.

For any $H$, the group $P(H)$ has an $R$-module structure *and* is acted upon by $F$.

Thus, $H$ is an $R[F]$-module, where $R[F]$ is the non-commutative ring of polynomials with

$$Fr = r^p F$$

for all $r \in R$.

Now suppose $\phi : H_1 \to H_2$ is a Hopf algebra homomorphism. Then $\phi(P(H_1)) \subseteq \phi(P(H_2))$. Furthermore, for $t \in P(H_1)$,

$$F\phi(t) = \phi(Ft),$$

so $\phi$ commutes with $F$.

Thus, $\phi\Big|_{P(H_1)}$ is an $R[F]$-module map.

Therefore, we have a functor

$$\{R\text{-Hopf alg.}\} \to \{R[F]\text{-modules}\}$$
$$H \to P(H)$$

which is not a categorical equivalence.

**Exercise 7.** Show that $RC_p^2$ and $RC_{p^2}$ correspond to the same $R[F]$-module.

**Exercise 8.** Show that the $R[F]$-module $(R[F])[X]$ does not correspond to any Hopf algebra $H$.

We will get a categorical equivalence by restricting the objects in both categories.

# Outline

We say $H$ is *primitively generated* if no proper subalgebra of $H$ contains $P(H)$.

Famous examples:

1. $H = R[t]/(t^p), \ t \in P(H)$.
2. $H = R[t]/(t^{p^n}), \ t \in P(H)$.
3. $H = R[t_1, \dots, t_n]/(t_1^{p^{r_1}}, \dots, t_n^{p^{r_n}}), \ t_i \in P(H)$.
4. $H = RC_p^*$
5. $H = R(C_p^n)^*$

**Exercise 9.** Let $H = R[t]/(t^{p^2})$ with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t^{pi} \otimes t^{p(p-i)}.$$

This is a Hopf algebra (not the exercise). Show that $H$ is not primitively generated (yes, the exercise).

From here on, we suppose $H$ is primitively generated and $R$ is a PID.

Then $P(H)$ is free over $R$ and of finite type over $R[F]$.

So the functor from before can restrict to

$$\left\{ \begin{array}{c} \text{primitively generated} \\ R\text{-Hopf algebras} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} R[F]\text{-modules of finite type,} \\ \text{free over } R \end{array} \right\}.$$

We claim this is a categorical equivalence.

# $H \mapsto P(H)$

We will construct an inverse.

Let $M$ be an $R[F]$-module of finite type, free over $R$.

Let $\{e_1, e_2, \ldots, e_n\}$ be an $R$-basis for $M$.

For $1 \leq i, j \leq n$, pick $a_{i,j} \in R$ such that

$$Fe_i = \sum_{j=1}^{n} a_{j,i} e_j.$$

Let

$$H = R[t_1, t_2, \ldots, t_n]/\{(t_i^p - \sum_{j=1}^{n} a_{j,i} t_j)\}$$

with $t_i$ primitive for all $i$.

Then, after identifying $t_i$ with $e_i$, $P(H) = M$.

We will call any $R[F]$-module of finite type over $R$ a *Dieudonné module*.

We will write

$$D_*(H) = P(H)$$

**Question.** Why not simply call the functor $P(-)$?

- For clarity, so we may better differentiate $H$ from its Dieudonné module.
- To get the audience used to $D_*$, which will be used extensively over the next few days (and usually won't mean $P(-)$).

# Properties of $D_*$

1. $D_*$ is covariant (although this is typically referred to as an example of a "contravariant Dieudonné module theory").
2. If $G = \mathrm{Spec}(H)$ then the Dieudonné module is $M = \mathrm{Hom}_{R\text{-gr}}(G, \mathbb{G}_a)$ where $\mathbb{G}_a$ is the additive group scheme. So $\mathrm{Spec}(H) \mapsto D_*(H)$ is a contravariant functor.
3. $D_*$ is an exact functor.
4. $D_*(H_1 \otimes H_2) \cong D_*(H_1) \times D_*(H_2)$.
5. $\mathrm{rank}_R H = p^{\mathrm{rank}_R M}$
6. $D_*$ respects base change (explained later).

# Some examples

**Example**

Let $H = R[t]/(t^p),\ t$ primitive.
Then $D_*(H) = Re$ with $Fe = 0$.

**Example**

Let $H = R[t]/(t^{p^n})$, $t$ primitive.
Then $P(H)$ is generated as a free $R$-module by $\{t, t^p, t^{p^2}, \dots, t^{p^{n-1}}\}$.
If we write $e_i = t^{p^i}; 0 \leq i \leq n-1$ then

$$D_*(H) = \bigoplus_{i=0}^{n-1} Re_i,\ \ Fe_i = e_{i+1},\ \ Fe_{n-1} = 0$$

where $i < n - 1$.

# Some more examples

## Example

Let $H = RC_p^* = R[t]/(t^p - t),\ t$ primitive.
Then $D_*(H) = Re$ with $Fe = e$.

## Example

Let $M = Re_1 \oplus Re_2$ with

$$Fe_1 = e_2,\ Fe_2 = e_1.$$

Then the corresponding Hopf algebra is

$$H = R[t_1, t_2]/(t_1^p - t_2, t_2^p - t_1) \cong R[t]/(t^{p^2} - t)$$

with $t$ primitive.
Note that if $\mathbb{F}_{p^2} \subseteq R$ then $H = RC_{p^2}^*$.

# Outline

# Outline

Consider the following problem: for a fixed $n$, find all primitively generated $R$-Hopf algebras of rank $p^n$.

Equivalently, find all $R[F]$ modules which are free rank $n$ over $R$.

These can be described with matrices.

Let $M$ be a free $R$-module, basis $\{e_1, \ldots, e_n\}$. Pick a matrix $A \in M_n(R)$ and use it to define an $F$-action on $M$ by

$$Fe_i = Ae_i.$$

Then $M$ corresponds to a Hopf algebra which can be quickly computed.

Each element of $M_n(R)$ gives a Hopf algebra.

Conversely, each Hopf algebra gives a matrix.

However, different choices of matrices can give the same Hopf algebra.

It can be shown (as it was in Exeter) that $A, B \in M_n(R)$ give the same Hopf algebra if any only if there is a $\Theta \in M_n(R)^\times$ such that

$$B = \Theta^{-1} A \Theta^{(p)}, \text{ where } \Theta = (\theta_{i,j}) \Rightarrow \Theta^p = (\theta_{i,j}^p).$$

Thus, a parameter space for primitively generated Hopf algebras of rank $p^n$ is $M_n(R)/\sim$, where $\sim$ is the relation above.

Presumably, this allows for a count of Hopf algebras for certain $R$, e.g. $R$ is a finite field.

# Outline

Let $H$ be a primitively generated $R$-Hopf algebra, and let $\mathrm{End}(H)$ be the $R$-Hopf algebra endomorphisms of $H$.

If $\phi : H \to H$ is a map of Hopf algebras it induces a map $\Theta : M \to M$ of $R[F]$-modules, where $M = D_*(H)$. We view $\Theta$ as a matrix with respect to a choice of some basis for $M$.

Again, as $M$ is a free $R$-module, for the same choice of basis the action of $F$ is determined by a matrix $A$.

Then

$$\Theta A = A\Theta^{(p)}.$$

Thus, the elements in $\mathrm{End}(H)$ are in 1-1 correspondence with matrices in $M_n(R)$ such that the equality above holds.

The ones with $\Theta \in M_n(R)^\times$ are, of course, the automorphisms.

## Example

Let $H = R[t]/(t^{p^2}),\ t$ primitive. Then, assuming $D_*(H) = Re_1 \oplus Re_2$ where $e_1$ corresponds to $t$ and $e_2$ to $t^p$,

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then, if $\Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an $R[F]$-linear map,

$$\Theta A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}$$

$$A\Theta^{(p)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a^p & b^p \end{pmatrix}$$

so $d = a^p, b = 0$.

The corresponding $\phi : H \to H$ is given by $\phi(t) = at + ct^p$ for $a, c \in R$, clearly an isomorphism iff $a \in R^\times$.

# Exercises

**Exercise 10.** Use Dieudonné modules to describe $\mathrm{End}(RC_p^*)$.

**Exercise 11.** Use Dieudonné modules to describe $\mathrm{Aut}(RC_p^*)$.

**Exercise 12.** Use Dieudonné modules to describe $\mathrm{End}(R(C_p \times C_p)^*)$.

**Exercise 13.** Use Dieudonné modules to describe $\mathrm{Aut}(R(C_p \times C_p)^*)$.

# Outline

The theory of Dieudonné modules can be extended beyond finite $R$-Hopf algebras.

For example, $H = R[t], \Delta(t) = t \otimes 1 + 1 \otimes t$ corresponds to a free $R[F]$-module of rank 1.

This allows us to construct projective resolutions for Dieudonné modules, and extensions of Dieudonné modules.

# An example

Let $H = R[t]/(t^p)$, $M$ its Dieudonné module.

Let $D_*(R[t]) = R[F]e$ (a free $R[F]$ module of rank 1).

Multiplication by $F : R[F]e \to R[F]e$ gives the projective resolution

$$0 \to R[F]e \to R[F]e \to M \to 0$$

for $M$.

This gives rise to

$$0 \to \mathrm{Hom}_{R[F]}(R[F]e, M) \to \mathrm{Hom}_{R[F]}(R[F]e, M) \to \mathrm{Ext}^1_{R[F]}(M, M) \to 0.$$

$$0 \to \mathrm{Hom}_{R[F]}(R[F]e, M) \to \mathrm{Hom}_{R[F]}(R[F]e, M) \to \mathrm{Ext}^1_{R[F]}(M, M) \to 0.$$

We have $\mathrm{Hom}_{R[F]}(R[F]e, M) \cong M$ by $f \mapsto f(e)$, hence

$$0 \to M \to M \to \mathrm{Ext}^1_{R[F]}(M, M) \to 0.$$

The map $M \to M$ induced from $\cdot F$ is trivial (since $Fm = 0$ for all $m \in M$), giving

$$\mathrm{Ext}^1_{R[F]}(M, M) \cong M.$$

Explicitly, for $[f] \in \mathrm{Ext}^1_{R[F]}(M, M)$ arising from $f \in \mathrm{Hom}_{R[F]}(R[F]e, M)$, the extension $M_f$ obtained is $(R[F]e \times M)/I$, where

$$I = \{(Fx, f(x)) : x \in R[F]e\}.$$

Does $\mathrm{Ext}^1_{R[F]}(M, M)$ give all the Hopf algebra extensions?

**Exercise 14.** Prove that the answer is no. Hint: consider $H = R[t]/(t^{p^2})$ with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t^{pi} \otimes t^{p(p-i)}.$$

This $H$ is a Hopf algebra (still not the exercise).

**Note.** That the $H$ above is, in fact, a Hopf algebra will be obvious after tomorrow.

# Outline

# Base change

Let $S$ be a flat extension of $R$.

Then we have two Dieudonné module theories:

1. $R$-Hopf algebras to $R[F]$-modules. Denote this functor $D_{*,R}$.
2. $S$-Hopf algebras to $S[F]$-modules. Denote this functor $D_{*,S}$.

How are they related?

# Very closely

**Proposition**

*Let $H$ be a primitively generated $R$-Hopf algebra. Then $H \otimes_R S$ is a primitively generated $S$-Hopf algebra and*

$$D_{*,S}(H \otimes_R S) = D_{*,R}(H) \otimes_R S.$$

**Proof.**

Obvious, or

**Exercise 15.** Prove the above proposition. □

Let $R = K$ a field (still char. $p$), and let $H_1, H_2$ be primitively generated $K$-Hopf algebras.

Let $S = L$ be an extension of $K$.

Then $H_1$ and $H_2$ are $L$-forms of each other if and only if $H_1 \otimes_K L \cong H_2 \otimes_K L$ as $L$-Hopf algebras.

Alternatively, $H_1$ and $H_2$ are $L$-forms of each other if and only if $D_{*,L}(H_1 \otimes_K L) \cong D_{*,L}(H_2 \otimes_K L)$.

If $M_i = D_{*,L}(H_i),\ i = 1, 2$, then $H_1$ and $H_2$ are $L$-forms of each other if

$$LM_1 \cong LM_2$$

as $L[F]$-modules.

## Example

Let $H = \mathbb{F}_p[t]/(t^p - 2t)$.
Then $H \ncong \mathbb{F}_p C_p^*$ since this would imply the existence of an invertible $1 \times 1$ matrix $\Theta = (\theta),\ \theta \in \mathbb{F}_p$ such that

$$\theta \cdot 2 = \theta^p \quad (\text{“}\Theta A = B\Theta^{(p)}\text{”}),$$

in which case $\theta^{p-1} = 2$.
However, let $L = \mathbb{F}_{p^{p-1}}$. Then such a $\theta \in L$ exists.
So $H$ and $\mathbb{F}_p C_p^*$ are $L$-forms of each other.

# More exercises

**Exercise 16.** Show that $K[t]/(t^{p^n})$ has no non-trivial $L$ forms for any $L$.

**Exercise 17.** Let $H = \mathbb{F}_p[t_1, t_2]/(t_1^p - t_2, t_2^p - t_1)$. Determine, if possible, the smallest field $L$ such that $H$ and $(KC_{p^2})^*$ are $L$-forms.

**Exercise 18.** Let $M = D_*(H)$ for $H$ a $K$-Hopf algebra of rank $p^n$. Suppose $F$ acts freely on $M$. Show that $H$ and $(K\Gamma)^*$ are $K^{\mathrm{sep}}$-forms for some $p$-group $\Gamma$.

**Exercise 19.** Let $M = D_*(H)$ for $H$ a $K$-Hopf algebra of rank $p^n$. Suppose $F^r M = 0$ for some $r > 0$. Show that $H$ and $(K\Gamma)^*$ are not $K^{\mathrm{sep}}$-forms for any $p$-group $\Gamma$.

# Outline

Let $R = \mathbb{F}_q[[T]]$, $K = \mathbb{F}_q((T)),\ p \mid q$.

(This is more structure than necessary. For much of what follows, $R$ could be a DVR of characteristic $p$.)

Let $v_K$ be the valuation on $K$ with $v_K(T) = 1$.

We have $D_*$ compatible for the extension $R \to K$.

Pick $A, B \in M_n(R)$ and construct $R$-Dieudonné modules $M_A, M_B$ with $\operatorname{rank}_R M_A = \operatorname{rank}_R M_B = n$, where the action of $F$ is given by $A, B$ respectively for some fixed $R$-basis for each.

Write $M_A = D_*(H_A)$ and $M_B = D_*(H_B)$.

Now $KH_A \cong KH_B$ if and only if there is a $\Theta \in \mathrm{GL}_n(K)$ such that

$$\Theta A = B\Theta^{(p)} \text{ for some } \Theta \in \mathrm{GL}_n(K).$$

$$\Theta A = B\Theta^{(p)} \text{ for some } \Theta \in \mathrm{GL}_n(K)$$

Write $A = (a_{i,j}), B = (b_{i,j}), \Theta = (\theta_{i,j})$. Then

- $H_A$ is viewed as an $R$-Hopf algebra using $A$, i.e.

$$H_A = R[u_1, \dots u_n]/(\{u_i^p - \sum a_{j,i}u_j\}).$$

- $H_B$ is viewed as an $R$-Hopf algebra using $B$, i.e.

$$H_B = R[t_1, \dots t_n]/(\{t_i^p - \sum b_{j,i}t_j\}).$$

- $KH_B$ is viewed as a $K$-Hopf algebra in the obvious way.
- $H_B$ is viewed as an order in $KH_B$ in the obvious way.
- $H_A$ is viewed as an order in $KH_B$ through $\Theta$, i.e.

$$H_A = R\left[\left\{\sum_{j=1}^{n} \theta_{j,i}t_j : 1 \leq i \leq n\right\}\right] \subset KH_B$$

(apologies for the abuses of language & the re-use of an old slide)

# rank $p$

Recall that $K$-Hopf algebras of rank $p$ correspond to rank 1 Dieudonné modules over $K$.

Fix $b \in K$, and let $M = Ke$ with $Fe = be$.

The corresponding Hopf algebra is $H_b := K[t]/(t^p - bt)$.

Furthermore, $H_a \cong H_b$ if and only if there is a $\theta \in K^\times$ with $\theta a = b\theta^p$.

**Case** $b = 0$**.** This is the algebra $K[t]/(t^p)$.

**Case** $b \neq 0$**.** From the exercises, $H_b$ is an $K^{\text{sep}}$-form of $KC_p^*$.

# Case $b \neq 0$

Since $K[t]/(t^p - bt) \cong K[t]/(t^p - T^{p-1}bt)$ by the map $t \mapsto T^{-1}t$, we may assume

$$0 \leq v_K(b) < p - 1.$$

Pick $\theta \in K^\times$ and let $a = b\theta^{p-1}$.

Provided $a \in R$ (which holds iff $\theta \in R$) we have

$$H_\theta = R[\theta t] \subset K[t]/(t^p - bt).$$

If $u = \theta t$ then

$$u^p = (\theta t)^p = \theta^p t^p = \theta^{p-1} b\theta t = au$$

and so $H_\theta = R[u]/(u^p - au)$.

Here, $H_\theta$ depends only on $v_K(\theta)$, so a complete list is

$$H_i = R\left[T^i t\right], i \geq 0.$$

# Case $b = 0$

Clearly, $\theta a = b\theta^p$ can occur only if $a = 0$.

So, any two $R$-Hopf orders in $K[t]/(t^p)$ are isomorphic as Hopf algebras.

**However.** They are not necessarily the same Hopf order. It depends on the chosen embedding $\Theta = (\theta) \in GL_1(K)$.

Let $\theta \in K^\times$.

Then $R[\theta t]$ is a Hopf order in $K[t]/(t^p)$.

As $R[\theta t] = R[(r\theta)t], r \in R^\times$, the complete list is

$$H_i = R[T^i t], i \in \mathbb{Z}.$$

# Overview for all $n$

- Pick a $K$-Hopf algebra $H$, and find the $B \in M_n(K)$ which is used in the construction of its $K$-Dieudonné module.
- Find $A \in M_n(R)$ such that $\Theta A = B\Theta^{(p)}$ for some $\Theta \in \mathrm{GL}_n(K)$. (One such example: $A = B$, $\Theta = I$. )
- Construct the $R$-Dieudonné module corresponding to $A$.
- Construct the $R$-Hopf algebra $H_A$ corresponding to this Dieudonné module.
- The algebra relations on $H_A$ are given by the matrix $A$.
- $H_A$ can be viewed as a Hopf order in $H$ using $\Theta$.
- $H_{A_1} = H_{A_2}$ if and only if $\Theta^{-1}\Theta'$ is an invertible matrix in $R$, where

$$\Theta A_1 = B\Theta^{(p)} \text{ and } \Theta' A_2 = B(\Theta')^{(p)}.$$

Alternatively, $H_{A_1} = H_{A_2}$ if and only if $\Theta' = \Theta U$ for some $U \in M_n(R)^\times$.

# $\Theta A = B\Theta^{(p)}$

**One strategy.** Given $B$, set $A = \Theta^{-1}B\Theta^{(p)}$.

This will generate a Hopf order iff $A$ has coefficients in $R$.

**But,** we can replace $\Theta$ with $\Theta U$ for $U \in M_2(R)^\times$.

Notice that

$$(\Theta U)^{-1}B(\Theta U)^{(p)} = U^{-1}\left(\Theta^{-1}B\Theta^{(p)}\right)U^{(p)},$$

and so $(\Theta U)^{-1}B(\Theta U)^{(p)} \in M_n(R)$ iff $\Theta^{-1}B\Theta^{(p)} \in M_n(R)$.

**But,** we can replace $\Theta$ with $\Theta U$ for $U \in M_2(R)^\times$.

## Theorem (New!)

*Let $H$ be a primitively generated $K$-Hopf algebra of rank $p^n$, and let $B$ be the matrix associated to $H$. Let $H_0$ be an $R$-Hopf order in $H$, and let $A$ be the matrix associated to $H_0$. Then there exists $\Theta = (\theta_{i,j}) \in GL_n(K)$ such that*

1. *$\Theta$ is lower triangular,*
2. $v_K(\theta_{i,i}) \geq v_K(\theta_{i,j}),\ j \leq i \leq n$,
3. $\theta_{i,i} = T^{v_K(\theta_{i,i})}$,
4. $\Theta A = B\Theta^{(p)}$.

In the case $n = 2$, the Hopf orders are of the form

$$H_{i,j,\theta} = R\left[T^i t_1 + \theta t_2, T^j t_2\right]$$

with $v_K(\theta) \leq j$.

**Q.** When is $H_{i,j,\theta} = H_{i',j',\theta'}$?

Precisely when there is a $U \in M_n(R)^\times$ such that

$$\begin{pmatrix} T^i & 0 \\ \theta & T^j \end{pmatrix} = \begin{pmatrix} T^{i'} & 0 \\ \theta' & T^j \end{pmatrix} U.$$

Such a $U$ exists if and only if

$$\begin{aligned} i &= i' \\ j &= j' \\ v_K(\theta - \theta') &\geq j. \end{aligned}$$

# An example: $H = K[t]/(t^{p^2}) = K[t_1, t_2]/(t_1^p, t_2^p - t_1)$

$$A = \begin{pmatrix} \theta^p T^{-i} & T^{pj-i} \\ -\theta^{p+1} T^{-(i+j)} & \theta T^{(p-1)j-i} \end{pmatrix}$$

To give a Hopf order, we require $pj \geq i$ and

$$v_K(\theta) \geq \min\{i/p, (i+j)/(p+1), i-(p-1)j\},$$

which can be simplified significantly, giving

$$H_{i,j,\theta} = R\left[T^i t_1 + \theta t_2, T^j t_2\right] = R\left[T^i t^p + \theta t, T^j t\right],$$
$$pj \geq i, \ i-(p-1)j \leq v_K(\theta) \leq j.$$

**Note.** $H_{i,j,\theta}$ is monogenic if and only if $pj = i$ and $v_K(\theta) = j$.

More exercises. All Hopf algebras below are primitively generated.

**Exercise 20.** Find all Hopf orders in $H = K[t_1, t_2]/(t_1^p, t_2^p)$.

**Exercise 21.** Find all Hopf orders in $H = K[t_1, t_2]/(t_1^p, t_2^p - t_2)$.

**Exercise 22.** Find all Hopf orders in $H = (KC_p^2)^*$.

**Exercise 23.** Determine which of the Hopf orders in $(KC_p^2)^*$ are monogenic.

**Exercise 24.** Find all Hopf orders in $H = K[t_1, t_2]/(t_1^p - t_2, t_2^p - t_1)$.

**Exercise 25.** Determine which of the Hopf orders in the previous problem are monogenic.

# Outline

1. Overview
2. Primitive Elements
3. A Dieudonné Correspondence
4. Applications
   - Classification of Hopf algebras
   - Endomorphisms of Hopf algebras
   - Extensions of Hopf algebras
   - Forms of Hopf algebras
   - Hopf orders
5. Want more?

# Let's assume you do.

There is a theory which is dual to that of primitively generated Hopf algebras.

In other words, a theory to classify $H$, where $H^*$ is primitively generated.

This would include elementary abelian group rings.

As far as I can figure out, given such an $H$, define

$$D_*(H) = H^+/(H^+)^2, \; H^+ = \ker \varepsilon.$$

It turns out that $D_*(H) = (D_*(H^*))^* := \mathrm{Hom}_R(D_*(H^*), R)$.

I don't know how the correspondence works without passing to duals.

Thank you.